620-295 Real Analysis with Applications

Assignment 4: Due 5pm on 18 September

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Due 5pm on 18 September in the appropriate assignment box on the ground floor of Richard Berry.

1. Define the following and give an example for each:
   (a) metric space,
   (b) complete (for a metric space),
   (c) completion (of a metric space),

2. Let X be a metric space and let (a_n) be a sequence in X. Show that if (a_n) converges then (a_n) is Cauchy.

3. Let (a_n) be a sequence in $\mathbb{R}_{\geq 0}$. Show that if $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

4. Prove that $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$ and $\lim_{x\to 0} \frac{\cos x - 1}{x^2} = -\frac{1}{2}$.

5. Define the following and give an example for each:
   (a) metric space,
   (b) limit of f as x approaches a,
   (c) limit of (x_n) as $n \to \infty$,
   (j) continuous at $x = a$,
   (c) continuous,
   (d) uniformly continuous,
   (e) Lipschitz,
   (f) ε-ball,

6. Define the following and give an example for each:
   (a) topology,
   (b) topological space,
   (c) open set,
   (d) closed set,
   (e) interior,

(f) closure,
(g) interior point,
(h) close point,
(i) neighborhood.

7. Let X and Y be metric spaces. Define the topology on X and Y. Define carefully what it means for $f : X \to Y$ to be continuous as a function between metric spaces and define carefully what it means for $f : X \to Y$ to be continuous as a function between topological spaces.

8. Let $a, b \in \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be a function. Let $c \in [a, b]$ and carefully define $f'(c)$. Prove that if $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are functions then $(fg)'(c) = f(c)g'(c) + f'(c)g(c)$, whenever $f'(c)$ and $g'(c)$ exist.